Continuous version of the Choquet Integral Representation Theorem

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Abstract

Let E be a locally convex topological Hausdorff space, K — its nonempty compact convex subset, μ — a regular, probability Borel measure on E and $\gamma > 0$. We say that the measure μ γ — represents point $x \in K$, if for any $f \in E^*$ an inequality $\sup_{\|f\|\leq 1} |f(x) - \int_K f d\mu| < \gamma$ holds. In this paper the continuous version of the Choquet theorem is proved. Namely, it is shown that if P is continuous multivalued mapping from a metric space T into the space of nonempty, bounded convex subsets of a Banach space X, then there exists a weak* continuous family (μ_t) of regular Borel probability measures on X γ – representing points in $P(t)$. The two cases are considered: in the first one the values of P are compact while in the second — closed. For this purpose it is shown (using geometrical tools) that the mapping $t \to \text{ext}P(t)$ is lower semicontinuous. The continuous versions of the Krein – Milman theorem are obtained as corollaries.

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1 Introduction

The classical Minkowski – Carathéodory representation theorem states that each point of a compact convex set K in $\mathbb{R}^n$ can be written as a convex combination of at most $n+1$ extreme points of K. This theorem was generalized by G. Choquet ([5]) who proved that each k point of a compact, convex and metrizable subset K of a locally convex Hausdorff topological space X is a barycenter of a regular Borel probability measure μ_k on X, supported by extreme points points of K, i.e. that the equality

$$f(k) = \int_K f d\mu_k$$

holds for any $f \in X^*$ with $\mu_k(\text{ext}K) = 1$, where ext$K$ stands for the set of extreme points of K.

E. Bishop and K. de Leeuw ([3]) removed the metrizability assumption. G. A. Edgar ([7]) proved the noncompact version of Choquet theorem. His result stated that the thesis of Choquet – Bishop – de Leeuw remained true for K being nonempty bounded, closed, convex and separable subset of a Banach space X having Radon – Nikodým property (RNP for short). In the paper [8] he improved his result by removing separability condition. P.Mankiewicz ([11]) modified it by introducing “separable extremal ordering”, more natural and easy to use partial order than the one introduced by Edgar (see [4], p.174).

The purpose of this paper is to show that an analogue of the Choquet theorem holds for ”moving” sets. These sets are values of a multivalued mapping, defined on a metric space T into suitable subsets of a certain Banach space X. We shall consider two cases.

In the first one multivalued mapping $P\colon T \rightsquigarrow X$ is continuous with compact convex values in a separable Banach space X. In the second case X is separable, reflexive Banach space while $P\colon T \rightsquigarrow X$ is continuous with bounded closed convex values (recall that reflexive Banach space has RNP). By the celebrated Michael theorem there exists a continuous function $p\colon T \to X$, called a continuous selection of P, with the property

that $p(t) \in P(t)$ for all $t \in T$. It will be shown that each such point $p(t)$ is, for given selection $p(\cdot)$, an „almost barycenter" of the regular Borel probability measure μ_t on X such that $\mu(\mathrm{ext}P(t)) = 1$. In other words, for any continuous selection $p(t)$ of the multifunction $P(t)$ there exists a continuous (in the weak* topology) family of measures $(\mu_t)_{t\in T}$ „almost representing" points $p(t)$. In both cases the fact that multifunction

$$t \to \mathrm{ext}P(t)$$

is lower semicontinuous is crucial for the main result. As obvious corollaries we obtain continuous versions of the Krein – Milman theorem.

All the necessary information about multifunctions can be found in [9], for Choquet theorem see [1], [13] and [4] (noncompact case), Banach spaces with Radon – Nikodym property are subject of the classics [4], [6], properties of measures on metric spaces are investigated in [2] (where one can also find a chapter devoted to multivalued mappings).

2 Preliminaries

In this section we state several definitions and facts needed.

Definition 2.1 *Let X and Y be topological spaces and $P\colon X \rightsquigarrow Y \setminus \emptyset$ — a set – valued map. We say that*

(a) *P is lower semicontinuous (lsc) iff the set*

$$P^{-}(U) := \{x \in X : P(x) \cap U \neq \emptyset\}$$

is open whenever $U \subset Y$ is open;

(b) *P is upper semicontinuous (usc) iff the set*

$$P^{-}(V) := \{x \in X : P(x) \subset U\}$$

is closed whenever $V \subset Y$ is closed;

(c) *P is continuous iff it is both lower – and upper semicontinuous.*

Theorem 2.1 *(Michael) Let X be a paracompact space, Y – a Banach space and $P\colon X \rightsquigarrow Y$ – lower semicontinuous multifunction with convex values. Then:*

(a) *for any $\varepsilon > 0$ there exists a continuous function $p_\varepsilon\colon X \to Y$ such that $d(p_\varepsilon(x), P(x)) < \varepsilon\ \forall x \in X$; this function is called an ε – selection of P;*

(b) *if in addition the values of P are closed, then there exists a continuous function $p\colon X \to Y$ such that $p(x) \in P(x)$; this function is called a continuous selection of P.*

Now denote by X a locally convex topological Hausdorff space and let K be its compact convex subset. If μ is a regular, Borel probability measure on X, we say that it is supported by the set $A \subset X$ (not necessarily closed) if $\mu(A) = 1$.

Definition 2.2 *For such X, K and μ be as above. We say that:*

(a) *measure μ represents point $x \in K$, if for all $f \in X^*$ we have*

$$f(x) = \int_K f d\mu.$$

This point, denoted by $r(\mu)$, is called the barycenter of μ;

(b) *measure μ γ — represents point $x \in K$, $\gamma > 0$, if for any $f \in X^*$ an inequality*

$$\sup_{\|f\| \leq 1} \left| f(x) - \int_K f d\mu \right| < \gamma.$$

holds.

Theorem 2.2 *(Choquet) Let X, K and μ be as above and assume additionally that K is metrizable. Then for any $x \in K$ there exists a regular Borel probability measure μ_x representing point x and such that $\mu_x(\text{ext}K) = 1$.*

Recall that if the set $\text{ext}K$ is closed then the Choquet theorem is equivalent to the Krein – Milman theorem.

Theorem 2.3 *(Edgar – Mankiewicz noncompact version of the Choquet theorem) Let K be a (nonempty) closed bounded convex subset of a Banach space X and suppose that K has RNP. Then every point of K is a barycenter of a regular Borel probability measure on K supported by the set* $\text{ext}K$.

3 Compact case

In this section we deal with multifunction P from T into compact convex sets of X.

We first establish the lower semicontinuity of map with values in the set of extreme points of the compact convex set. Recall that exposed point of a compact convex subset of a Banach space is a strongly exposed point of this subset.

Proposition 3.1 *Let T be a metric space, X — a Banach space, $P\colon T \rightsquigarrow X$ — a continuous multifunction with compact convex values. Then the multifunction*

$$t \to \text{ext}P(t)$$

is lower semicontinuous.

Proof. Let (t_n) be a sequence in T, convergent to a point $t_0 \in T$. It is enough to show that for each such sequence and any $a_0 \in \text{ext}P(t_0)$ there exists sequence (a_n), $n \in \mathbb{N}$, such that $a_n \in \text{ext}P(t_n)$ and $a_n \xrightarrow[n\to\infty]{} a_0$.

Let e_0 be any exposed (in fact strongly exposed) point of $P(t_0)$. Then there exists functional $f_0 \in X^*$, with unit norm, strongly exposing e_0. The lower semicontinuity of P yields existence of a sequence $(x_n) \subset X$, convergent to e_0, with $x_n \in P(t_n)$. Fix a number $\gamma > 0$ and define the slice

$$R_\gamma(t_n) := \{x \in P(t_n) : f_0(x) > c(f_0, P(t_n)) - \gamma\},$$

where $c(\cdot, A)$ stands for the support function of the set A. It turns out that there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the intersection $R_\gamma(t_n) \cap \text{ext}P(t_n)$ is nonempty. Suppose not. Then for each $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ for which $R_\gamma(t_n) \cap \text{ext}P(t_n) = \emptyset$, i.e. $\text{ext}P(t_n) \subset X \setminus R_\gamma(t_n)$. This implies the existence of a subsequence $n_k \xrightarrow[k\to\infty]{} \infty$ having property that for each $e \in \text{ext}P(t_{n_k})$

$$f_0(e) < c(f_0, P(t_{n_k})) - \gamma.$$

By the Krein – Milman theorem the set $P(t_{n_k})$ coincides with the closed convex hull of its extreme points, so in particular we have

$$f_0(x_{n_k}) \leq c(f_0, P(t_{n_k})) - \gamma, \quad k = 1, 2, ...$$

Passing to the limit we obtain the inequality

$$f_0(e_0) \leq c(f_0, P(t_0)) - \gamma,$$

contradicting the fact, that e_0 is strongly exposed.

Now let $\gamma = \frac{1}{m}$, $m = 1, 2, ...$, and consider slices $R_{\frac{1}{m}}(\cdot)$. Then for each m there exists n_m such that for $n \geq n_m$ we have

$$R_{\frac{1}{m}}(t_n) \cap \text{ext}P(t_n) \neq \emptyset.$$

We can assume that $n_m \leq n < n_{m+1}$. For such n choose $e_n \in R_{\frac{1}{m}}(t_n) \cap \text{ext}P(t_n)$ obtaining the sequence (e_n) with the property

$$f_0(e_n) \geq c(f_0, P(t_n)) - \tfrac{1}{m}$$

for $n_m \leq n < n_{m+1}$.

By the upper semicontinuity of P there exists subsequence of (e_n) (denoted also (e_n)) convergent to a point $\overline{e}_0 \in P(t_0)$. As the values of P are convex sets, we can use the relationship between the Hausdorff distance $h(A, B)$ of the sets A, B and their support functions, so

$$\sup_{\|f\| \leq 1} \{|c(f, P(t_n)) - c(f, P(t_0))|\} = h(P(t_n), P(t_0)).$$

The compactness of $P(t)$, $t \in T$, yields the continuity of P in Hausdorff metric, which gives us $h(P(t_n), P(t_0)) \xrightarrow[n\to\infty]{} 0$, which in turn implies that for any $f \in X^*$ $c(f, P(t_n)) \xrightarrow[n\to\infty]{} c(f, P(t_0))$. But for $n_m \leq n \leq n_{m+1}$ we have $f_0(e_n) \geq c(f_0, P(t_n)) - \frac{1}{m}$. Taking into account that $f_0(e_n)$ converges to $f_0(\overline{e})$ we finally get $e = \overline{e}_0$.

Now by Lindenstrauss – Troyanski result ([10], [15], see also [4]) the set $P(t_0)$ equals the closed convex hull of its (strongly) exposed points and by Milman partial converse of the Krein – Milman theorem those points are dense in the set $\text{ext}P(t_0)$.

We are now ready to construct a desired sequence of extreme points. So let a_0 be any extreme point of $P(t_0)$. Choose and fix $n_1 \in \mathbb{N}$ and $e_0^{n_1} \in (\text{st})\text{exp}P(t_0)$. There exists sequence (b_n^1) of extreme points of $P(t_0)$ convergent to $e_0^{n_1}$. Then there exists $n_2 > n_1$ such that for $n \geq n_2$ we have $\|e_0^{n_1} - b_n^1\| < \frac{1}{n_1}$. Now take $e_0^{n_2} \in (\text{st})\text{exp}P(t_0)$ with $\|e_0^{n_2} - a_0\| < \frac{1}{n_2}$ and sequence (b_n^2) of extreme points of $P(t_n)$ convergent to $e_0^{n_2}$. Then there exists such $n_3 > n_2$ that for all $n \geq n_3$ we have the inequality $\|e_0^{n_2} - b_n^2\| < \frac{1}{n_1}$. Repeating this procedure we obtain sequences (b_n^i), $b_n^i \in \text{ext}P(t_n)$. Setting $a_n := b_n^i$ we obtain the desired sequence. ■

Remark. Tolstonogov and Figonienko proved (under the same assumptions) in [14] the lower semicontinuity of the map $t \to \text{clext}P(t)$, where „cl" stands for „closure". This result is equivalent to the above one, but the method of the proof presented here is of geometrical nature, in contrast to the topological methods they used. Incidentally, it seems that considering the map $t \to \text{ext}P(t)$, instead of $t \to \text{clext}P(t)$ is „in the spirit" of the Choquet theorem.

Now let T be a metric space, X — separable Banach space. By $\mathcal{M}(X)$ we denote regular probability Borel measures on X. We consider a continuous multifunction

$$P \colon T \rightsquigarrow X,$$

with (nonempty) compact convex values. The Michael selection theorem assures us of the existence of a continuous selection p of P. Define the set – valued map $L \colon T \rightsquigarrow \mathcal{M}(X)$ setting

$$L(t) := \left\{ \mu \in \mathcal{M}(X) : \mu(\text{ext}P(t)) = 1, \quad \sup_{\|f\|\leq 1} \left| f(p(t)) - \int\limits_{P(t)} f d\mu \right| < \gamma, \quad f \in X^* \right\},$$

where γ is fixed positive number. Choquet theorem guarantees that $L(t)$ is nonempty for all $t \in T$. Choose and fix the continuous selection p of P.

Proposition 3.2 *Multifunction L is lower semicontinuous.*

Proof. It is enough to show that for any sequence (t_n) of T, convergent to $t_0 \in T$ and for any nonempty, weakly* closed subset F of $\mathcal{M}(X)$, an implication $L(t_n) \subset F \Rightarrow L(t_0) \subset F$ holds.

Take any element μ_0 of $L(t_0)$. The set of discrete measures on $\mathrm{clext}P(t_0)$ is dense in the set of measures supported by that set, so there exists a sequence (m_k) of discrete measures (i.e. convex combinations of Dirac measures), convergent weakly* to μ_0. This yields the existence of such k_0 that for all $k \geq k_0$ measure m_k $\gamma-$represents point $p(t_0)$. Each measure m_k, $k \geq k_0$, is of the form

$$m_k = \sum_{i=1}^{m} \lambda_i \delta_{a_i},$$

where $\delta_{(\cdot)}$ is Dirac measure, $a_i \in \mathrm{ext}P(t_0)$ and λ_i are coefficients of the convex combination. As $\mathrm{ext}P(\cdot)$ is lower semicontinuous, for any a_i, $i = 1, 2, ..., m$, there exists a sequence (b_n^i), $b_n^i \in \mathrm{ext}P(t_n)$, convergent to a_i. This means that for fixed k the sequence (μ_n^k) of measures, $\mu_n^k = \sum\limits_{i=1}^{m} \lambda_i \delta_{b_n^i}$ converges to m_k. For $f \in X^*$ we also have

$$\begin{aligned}
\left| f(p(t_n)) - \int\limits_{P(t_n)} f d\mu_n^k \right| \leq & \\
& \leq |f(p(t_n)) - f(p(t_0))| + \left| f(p(t_0)) - \int\limits_{P(t_0)} f d\mu_0 \right| + \\
& + \left| \int\limits_{P(t_0)} f d\mu_0 - \int\limits_{P(t_0)} f dm_k \right| + \left| \int\limits_{P(t_0)} f dm_k - \int\limits_{P(t_n)} f d\mu_n^k \right|.
\end{aligned}$$

The first and the last terms on the right converge to zero. Since there exists $M > 0$ such that $\sup\{\|y\| : y \in P(t_0)\} \leq M$, so for $x \in P(t_0)$ we have $|f(x)| \leq M|f(\frac{1}{M}x)| \leq M$. Taking into account that the sets $\mathrm{supp}m_k$ and $\mathrm{supp}\mu_0$ are the subsets of $P(t_0)$ we obtain

$$\sup_{\|f\|\leq 1} \left| \int\limits_{P(t_0)} f dm_k - \int\limits_{P(t_0)} f d\mu_0 \right| = \sup_{\|f\|\leq 1} \left| \int\limits_{P(t_0)} f d(m_k - \mu_0) \right| \leq M|(m_k - \mu_0)(1)| \xrightarrow[k\to\infty]{} 0. \tag{1}$$

For fixed $k \geq k_0$ and for all $n \geq n_0$ we then have

$$\left| f(p(t_n)) - \int\limits_{P(t_n)} f d\mu_n^k \right| < \gamma.$$

By construction, measure μ_n^k is supported by the set of extreme points of $P(t_n)$, so for $n \geq n_0$ it belongs to $L(t_n)$ and thus to F. Passing to the limit gives inclusion $\mu_0 \in F$ resulting the lower semicontinuity of the multifunction L. ■

Corollary 3.1 *Multifunction*

$$\mathrm{cl}L(t) = \left\{ \mu \in \mathcal{M}(X) : \mu(\mathrm{clext}P(t)) = 1, \sup_{\|f\|\leq 1} \left| f(p(t)) - \int\limits_{P(t)} f d\mu \right| < \gamma, \quad f \in X^* \right\}$$

is lower semicontinuous.

Corollary 3.2 *There exists a continuous selection* $\bar{l}\colon T \to \mathcal{M}(X)$ *of the multifunction* $\mathrm{cl}L$.

The set of extreme points of even compact convex set need not be closed, so in general the values of the multifunction L are not closed sets. In particular we cannot expect L to have continuous selections. However, there exists continuous approximate selections, as stated in the next result.

By $\mathcal{H}$ we will denote the Hilbert cube, $\mathcal{M}(\mathcal{H})$ stands for the regular probability Borel measures on $\mathcal{H}$; this is compact separable metric space.

Theorem 3.1 *(continuous version of Choquet Theorem)*
Let T be a metric space and X — separable Banach space. Consider continuous multifunction $P\colon T \rightsquigarrow X$ with compact convex values and denote by p its continuous selection. Consider also multifunction $L\colon T \rightsquigarrow \mathcal{M}(X)$:

$$L(t) := \left\{ \mu \in \mathcal{M}(X) : \mu(extP(t)) = 1, \ \sup_{\|f\|\le 1} \left| f(p(t)) - \int\limits_{P(t)} f d\mu \right| < \gamma, \ \ f \in X^* \right\}.$$

Then for any $\delta > 0$ L admits a continuous $\delta -$ selection of L.

Proof. In what follows we adopt classical Michael method. There exists (see [2], pp. 483 – 485) continuous mapping $\hat{\phi}\colon \mathcal{M}(X) \to \mathcal{M}(\mathcal{H})$. The Polish space $\mathcal{M}(X)$ is metrizable by

$$d(\mu_1, \mu_2) = \rho_{\mathcal{H}}(\hat{\phi}(\mu_1), \hat{\phi}(\mu_2)) = \sum_{j=1}^{\infty} \frac{1}{2^j} |(\hat{\phi}(\mu_1) - \hat{\phi}(\mu_2))(\zeta_j)|,$$

where ζ_j, $j = 1, 2, ...$, are from the dense set in $C(\mathcal{H})$ – the space of the continuous functions on the Hilbert cube. We start with fixing: a dense set $(\zeta_n)_{n=1}^{\infty}$ in $C(\mathcal{H})$, numbers $\delta > 0$ and $N \in \mathbb{N}$ with $\sum\limits_{j=N+1}^{\infty} \frac{1}{2^j} < \frac{\delta}{4}$, functions $\zeta_1, ..., \zeta_n \in (\zeta_n)_{n=1}^{\infty}$, point $t_0 \in T$ and measure $\mu_o \in L(t_0)$. The mapping $\mu \to |(\hat{\phi}(\mu) - \hat{\phi}(\mu_0))(\zeta_j)|$ is continuous for (any) μ_0 and any fixed $j \in \mathbb{N}$, so the set

$$V(\mu_0, \zeta_1, \ldots, \zeta_N, \delta) := \left\{ \mu : \sum_{j=1}^{N} \frac{1}{2^j} \left| (\hat{\phi}(\mu) - \hat{\phi}(\mu_0))(\zeta_j) \right| < \frac{\delta}{2} \right\}$$

is open. The lower semicontinuity of L implies that

$$U(t_0, \mu_0) := L^{-1}(V) = \{t \in T : L(t) \cap V \neq \emptyset\}$$

is an open neighborhood of t_0 is nonempty and open. Using the lower semicontinuity of L again we obtain an open cover $\{U(t_\alpha, \mu_\alpha)\}_{\alpha \in I}$ of T. By $e_\alpha(\cdot)$ denote locally finite partition of unity subordinated to this covering. Our candidate for continuous $\delta -$ selection l_δ of L is of the form

$$l_\delta(t) := \sum_{\alpha \in I} e_\alpha(t) \mu_\alpha.$$

Fix $t \in T$ and set $\{\alpha : e_\alpha(t) > 0\} := \{\alpha_1, ..., \alpha_k\}$; then $t \in \mathrm{supp} e_{\alpha_i} \subset U(t_{\alpha_i}, \mu_{\alpha_i})$, so the intersection of $L(t)$ with ball of radius δ, centered in μ_{α_i}, is nonempty. Now take measure $\overline{\mu}_i$ – the element of this intersection, and observe that $d(\overline{\mu}_i, \mu_{\alpha_i}) < \delta$. The point $\overline{l}_\delta(t) := \sum\limits_{i=1}^{k} e_{\alpha_i}(t) \overline{\mu}_i$ lies in the convex set $L(t) \cap V$. The thesis

of the theorem follows from the following sequence of inequalities:

$$d(l_\delta(t), L(t)) \le d(l_\delta(t), \bar{l}_\delta(t)) = \rho_{\mathcal{H}}(\hat{\phi}(l_\delta(t)), \hat{\phi}(\bar{l}_\delta(t))) \le$$

$$\le \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \left(\sum_{\alpha \in I} e_\alpha(t) \hat{\phi}(\mu_\alpha) - \sum_{i=1}^{k} e_{\alpha_i}(t) \hat{\phi}(\overline{\mu}_i) \right) (\zeta_j) \right| \le$$
$$\le \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{i=1}^{k} e_{\alpha_i}(t) \left| (\hat{\phi}(\mu_{\alpha_i}) - \hat{\phi}(\overline{\mu}_i))(\zeta_j) \right| \le$$
$$\le \sum_{j=1}^{N} \frac{1}{2^j} \sum_{i=1}^{k} \left| (\hat{\phi}(\mu_{\alpha_i}) - \hat{\phi}(\overline{\mu}_i))(\zeta_j) \right| + \sum_{j=N+1}^{\infty} \frac{1}{2^j} <$$
$$< \sum_{i=1}^{k} e_{\alpha_i}(t) \left(\sum_{j=1}^{N} \left| (\hat{\phi}(\mu_{\alpha_i}) - \hat{\phi}(\overline{\mu}_i))(\zeta_j) \right| \right) + \frac{\delta}{4} <$$
$$< \frac{\delta}{2} + \frac{\delta}{4} < \delta.$$

■

Corollary 3.3 *(continuous version of the Krein – Milman theorem)*
Let T, X, P and p be as above. Then for any $\gamma > 0$ there exists continuous family of measures $(\mu_t)_{t \in T}$ on X, supported by the closure of the set of extreme points of $P(t)$ and γ – representing point $p(t)$.

4 Noncompact case

In this section we consider the noncompact case. We have to impose more assumptions both on the Banach space X and multifunction P.

Recall that $x \in K$ is a denting point iff for each $\varepsilon > 0$ $x \notin \mathrm{clcv}(K \setminus U_\varepsilon(x))$, where $U_\varepsilon(x)$ denotes the ε — neighbourhood of x.

Theorem 4.1 *Let T be a metric space, X – reflexive Banach space and consider multifunction $P\colon T \rightsquigarrow X$ fulfilling the following conditions:*

(a) *P is continuous;*

(b) *for each $t \in T$ the set $P(t)$ is bounded, closed and convex;*

(c) *each extreme point of the set $P(t)$ is its denting point.*

Then the multifunction

$$t \to \mathrm{ext}P(t)$$

is lower semicontinuous.

Proof. Exactly as in the compact case we construct the slice $R_\gamma(\cdot)$ and show that it is nonempty, replacing " Krein – Milman theorem" with "Krein – Milman property", obtaining in the same way the sequence (e_n) of extreme points belonging for $n_n \le n \le n_{m+1}$ both to the set $\mathrm{ext}P(t_n)$ and the slice $R_{\frac{1}{m}}(t_n)$. By the upper semicontinuity of P this sequence is bounded and has a subsequence (denoted also by (e_n)) convergent to the point $\overline{e}_0 \in P(t_0)$. The upper semicontinuity of P implies its Hausdorff upper semicontinuity, so we can write

$$\sup\{c(f, P(t_n)) - c(f, P(t_0)) : \|f\| \le 1\} = h^*(P(t_n), P(t_0)) \xrightarrow[n\to\infty]{} 0,$$

where $h^*(A, B) = \sup\{d(a, B) : a \in A\}$. Thus $c(f, P(t_n)) \xrightarrow[n\to\infty]{} c(f, P(t_0))$ and (as in the compact case) we can conclude that $\overline{e}_0 = e_0$.

We have thus costructed the sequence (e_n), $e_n \in \mathrm{ext}P(t_n)$, weakly convergent to the point $e_0 \in \mathrm{stexp}P(t_0)$. Moreover, we have
$d(e_n, P(t_0)) \leq h^*(P(t_n), P(t_0)) \xrightarrow[n\to\infty]{} 0$, so there exists sequence (b_n) of elements of $P(t_0)$ with $\|b_n - e_n\| \xrightarrow[n\to\infty]{} 0$. This yields weak convergence of (b_n) to e_0 which in turn implies (as e_0 is strongly exposed), that (b_n) converges to e_0 in norm. Thus $\|e_n - e_0\| \xrightarrow[n\to\infty]{} 0$.

It turns out that the set of strongly exposed points of $P(t)$, $t \in T$, is dense in the set $\mathrm{ext}P(t)$. Indeed, suppose that this is not the case and consider slice of $P(t)$ with norm diameter ε containing some $e \in \mathrm{ext}P(t)$. None of the strongly exposed points of $P(t)$ belongs to the slice, so we have $P(t) = \mathrm{clcvstexp}P(t)$. This contradicts the fact that e is the denting point.

The rest of the proof proceeds as in the compact case. ■

Now we are able to reformulate continuous version of the Choquet theorem and its corollary.

Theorem 4.2 *(continuous version of the noncompact Choquet theorem)*
Let T be a metric space, X – separable, reflexive Banach space and consider multifunction $P\colon T \rightsquigarrow X$ fulfilling the following conditions:

(a) *P is continuous;*

(b) *for each $t \in T$ the set $P(t)$ is bounded, closed and convex;*

(c) *each extreme point of the set $P(t)$ is its denting point.*

Denote by p the continuous selection of P and define the set – valued map $L\colon T \rightsquigarrow \mathcal{M}(X)$:

$$L(t) := \left\{ \mu \in \mathcal{M}(X) : \mu(extP(t)) = 1, \sup_{\|f\|\leq 1} \left| f(p(t)) - \int_{P(t)} f d\mu \right| < \gamma, \quad f \in X^* \right\}.$$

Then for any $\delta > 0$ there exists continuous function $l_\delta\colon T \to \mathcal{M}(X)$, the $\delta -$ selection of L.

Corollary 4.1 *(continuous version of the noncompact Krein – Milman theorem)*
Let T, X, P and p be as above. Then for any $\gamma > 0$ there exists continuous family of measures $(\mu_t)_{t\in T}$ on X, supported by the closure of the set of extreme points of $P(t)$ and $\gamma -$ representing point $p(t)$.

The proofs of these results are identical to the ones given in the previous section.

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